

Parameter-dependent Pseudodifferential Operators of Toeplitz Type on Closed Manifolds

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ABSTRACT. We present a calculus of zero-order parameter-dependent pseudodifferential operators on a closed manifold M that contains both usual parameter-dependent operators—where the parameter enters as an additional covariable—as well as operators independent of the parameter. Parameter-ellipticity is characterized by the invertibility of three associated principal symbols. In case of ellipticity we can construct a parametrix that is an inverse for large values of the parameter. We then extend this parametrix-construction to operators of Toeplitz type, in particular, to operators of the form $P_1 A(\tau) P_0$ where both P_0 and P_1 are zero-order projections and $A(\tau)$ is a usual parameter-dependent operator of arbitrary order or $A(\tau) = \tau^\mu - A$ with a pseudodifferential operator A of positive order $\mu \in \mathbb{N}$ (in this case $P_0 = P_1$). Decay estimates in the parameter of the inverse can be derived.

1. Introduction

Over the past decades the theory of parameter-dependent pseudodifferential operators has proved to be an important tool for analyzing the existence and the structure of resolvents of differential or pseudodifferential operators. The general aim is to obtain a nice pseudodifferential structure of the resolvent of an operator A , and then to derive conclusions for example with respect to resolvent estimates (e.g. that A generates an analytic semigroup) and holomorphic functional calculi (e.g. that A admits a bounded H_∞ -calculus), the heat kernel e^{-tA} and heat trace asymptotics, the counting function, and many other arguments.

This idea has been realized for a large number of different kinds of differential or pseudodifferential operators on various kinds of manifolds, including boundary value problems and operators on manifolds with singularities. Let us illustrate the most elementary approach for operators on smooth closed manifolds: If M is a closed manifold and F_0, F_1 are two vector bundles over M let us denote by $L_{\text{cl}}^\mu(\mathbb{R}_+; M, F_0, F_1)$ the space of classical pseudodifferential operators of order $\mu \in \mathbb{R}$ acting from sections into F_0 to sections into F_1 and that depend on a parameter

$\tau \geq 0$ that enters as an additional co-variable (for simplicity of presentation we focus in this introduction on the case of a real parameter τ , while later we shall admit a more general parameter space). This means that in local coordinates x (with corresponding co-variable ξ) the local pseudodifferential symbols satisfy estimates of the form

$$|D_\xi^\alpha D_x^\beta D_\tau^k a(x, \xi, \tau)| \leq C_{\alpha\beta\gamma} (1 + |\xi| + |\tau|)^{\mu - |\alpha| - k},$$

Thus differentiation with respect to ξ or τ improves the decay in ξ and τ simultaneously. The phrase *classical*, indicated by the subscript cl, for us means that additionally a has a complete asymptotic expansion into components that are positively homogeneous with respect to (ξ, τ) ; see Section 2 for more details. With any such operator one can associate a homogeneous principal symbol which is a bundle homomorphism

$$\pi^* F_0 \longrightarrow \pi^* F_1, \quad \pi : (T^* M \times \overline{\mathbb{R}}_+) \setminus \{0\} \longrightarrow M,$$

where π is the canonical projection onto M . Due to the homogeneity this homomorphism is uniquely determined by its restriction to the unit-sphere bundle. In case of trivial bundles $F_j = M \times \mathbb{C}^{N_j}$ this homomorphism can be identified with a smooth function on $(T^* M \times \overline{\mathbb{R}}_+) \setminus \{0\}$ with values in the $(N_1 \times N_0)$ -matrices. An operator $A(\tau)$ is called *parameter-elliptic* if its homogeneous principal symbol is an isomorphism. In this case one can find a parametrix $B(\tau)$ of corresponding negative order that coincides with $A(\tau)^{-1}$ for sufficiently large values of τ . For example, $\tau^2 - \Delta$ with the Laplacian on M is a parameter-elliptic operator of order 2, and its parametrix/inverse is of order -2 . However, note that one cannot proceed in this way when the Laplacian is replaced by a pseudodifferential (and non-differential) operator A . This is due to the fact that $\tau^\mu - A$, $\mu = \text{ord } A$, is not a parameter-dependent operator in the above described sense. To cover this case Grubb in [Gr85] introduced a more general calculus of parameter-dependent operators (actually in her book a parameter-dependent version of Boutet de Monvel's algebra for boundary value problems is developed, containing operators on a closed manifold as a simple special case) that she later in [GS95] together with Seeley further refined for studying resolvent trace asymptotics for pseudodifferential operators and certain non-local boundary value problems.

The main motivation for the present paper was to study the invertibility of operators of the form $P_1 A(\tau) P_0$ for large values of τ and the structure of the inverse, where $A(\tau) \in L_{\text{cl}}^\mu(\overline{\mathbb{R}}_+; M, F_0, F_1)$ and $P_j \in L_{\text{cl}}^0(M, F_j, F_j)$ are two zero-order pseudodifferential projections not depending on the parameter. Using terminology introduced in [Sch01], we call such operators parameter-dependent operators of Toeplitz type. It seems that the approach of both [Gr85] and [GS95] does not apply in this situation, but can be suitably extended. In Section 4 we shall construct a calculus of pseudodifferential operators containing both parameter-dependent operators

in the above described sense as well as operators independent of the parameter. We focus on the zero-order case, since in the applications we make use of order reductions. The operators in this calculus do not have a homogeneous principal symbol defined on the unit-sphere bundle in $T^*M \times \overline{\mathbb{R}}_+$, but only defined outside the “north-poles” $(x, \xi, \tau) = (x, 0, 1)$, $x \in M$. However, they have a particular structure near each north-pole: In polar-coordinates they have (generalized) Taylor asymptotics; see Definition 4.10 for details. The parameter-ellipticity in our calculus is characterized by the invertibility of the homogeneous principal symbol outside the north-poles, the invertibility of the first Taylor term (a symbol on the co-sphere bundle of M), and the invertibility of a certain limit-family obtained as $\tau \rightarrow \infty$. Parameter-elliptic operators possess a parametrix within the calculus that coincides with the inverse for large values of the parameter. Once established this calculus we then employ the general approach to the analysis of Toeplitz pseudodifferential operators developed by the author in [Se11] (which we review in Section 3) and derive the corresponding notion of parameter-ellipticity for parameter-dependent Toeplitz operators.

As we shall discuss in Section 5 the calculus of Toeplitz operators permits us to treat operators of the form $P_1 A(\tau) P_0$ with $A(\tau) \in L_{\text{cl}}^\mu(\overline{\mathbb{R}}_+; M, F_0, F_1)$ and projections $P_j \in L_{\text{cl}}^0(M, F_j, F_j)$. Also operators of the form $P(\tau^\mu - A)P$ with $A \in L_{\text{cl}}^\mu(M, F, F)$ of positive integer order and a projection $P \in L_{\text{cl}}^0(M, F, F)$ are covered (though such operators can also be treated with the calculus of [Gru85]). In the present paper we focus on operators on closed manifolds but we hope to extend this concept in a forthcoming publication also to manifolds with boundary. In particular, this would allow us to treat the resolvent of operators like the Stokes operator and to develop an approach alternative to those used by Grubb as well as Giga [Gi81].

2. Important notations

In this section we shall introduce some notation that will be used throughout the paper. With some fixed choice of $0 \leq a \leq b < 2\pi$ we set

$$(2.1) \quad \Lambda = \Lambda(a, b) := \left\{ \lambda = (\tau, \theta) \mid \tau \geq 0, a \leq \theta \leq \tau \right\} \subset \mathbb{R}^2$$

(in applications Λ may also be identified with a sector in the complex plane via polar-coordinates).

2.1. Pseudodifferential symbols. With $\mu \in \mathbb{R}$ and $N_0, N_1 \in \mathbb{N}$ we let $S^\mu(\mathbb{R}^n, \mathbb{R}^n \times \Lambda; N_0, N_1)$ denote the space of all functions $a : \mathbb{R}_x^n \times \mathbb{R}_\xi^n \times \Lambda \rightarrow \mathbb{C}^{N_1 \times N_0}$ taking values in the complex $(N_1 \times N_0)$ -matrices that are continuous and infinitely many times continuously differentiable with respect to (x, ξ, τ) and satisfy uniform estimates

$$(2.2) \quad |D_\xi^\alpha D_x^\beta D_\tau^k a(x, \xi, \lambda)| \leq C_{\alpha\beta\gamma} \langle \xi, \tau \rangle^{\mu - |\alpha| - |\gamma|}$$

for any order of derivatives; as usual we have $\langle \xi, \tau \rangle = (1 + |\xi|^2 + |\tau|^2)^{1/2}$. The symbol a is called *classical* if there exists a sequence of symbols $a_{\mu-j} \in S^{\mu-j}(\mathbb{R}^n, \mathbb{R}^n \times \Lambda; N_0, N_1)$ which are homogeneous in the large of degree $\mu - j$, i.e.,

$$a_{\mu-j}(x, t\xi, t\tau, \theta) = t^{\mu-j} a(x, \xi, \tau, \theta) \quad \forall t \geq 1 \quad \forall |(\xi, \tau)| \geq 1,$$

such that

$$a - \sum_{j=0}^{\ell-1} a_{\mu-j} \in S^{\mu-j-\ell}(\mathbb{R}^n, \mathbb{R}^n \times \Lambda; N_0, N_1).$$

The functions

$$(2.3) \quad a^{(\mu-j)}(x, \xi, \tau, \theta) = a_{\mu-j}\left(x, \frac{\xi}{|(\xi, \tau)|}, \frac{(\tau, \theta)}{|(\xi, \tau)|}\right), \quad (\xi, \tau) \neq 0,$$

are called the homogeneous components of a and $a^{(\mu)}$ the *homogeneous principal symbol* of a . The space of classical symbols of order μ shall be denoted by $S_{\text{cl}}^{\mu}(\mathbb{R}^n, \mathbb{R}^n \times \Lambda; N_0, N_1)$.

We shall also use symbols without parameter. The classes $S^{\mu}(\mathbb{R}^n, \mathbb{R}^n; N_0, N_1)$ and $S_{\text{cl}}^{\mu}(\mathbb{R}^n, \mathbb{R}^n; N_0, N_1)$ are defined as above by eliminating everywhere the parameter λ .

With $a \in S^{\mu}(\mathbb{R}^n, \mathbb{R}^n; N_0, N_1)$ we associate the pseudodifferential operator $\text{op}(a) = a(x, D)$ defined by

$$[\text{op}(a)u](x) = \int e^{ix\xi} a(x, \xi) \widehat{u}(\xi) d\xi;$$

then $\text{op}(a)$ defines an operator $\mathcal{S}(\mathbb{R}^n, \mathbb{C}^{N_0}) \rightarrow \mathcal{S}(\mathbb{R}^n, \mathbb{C}^{N_1})$ between the spaces of rapidly decreasing functions that extends by continuity to a map between the standard Sobolev (Bessel potential) spaces, $\text{op}(a) : H^s(\mathbb{R}^n, \mathbb{C}^{N_0}) \rightarrow H^{s-\mu}(\mathbb{R}^n, \mathbb{C}^{N_1})$ for arbitrary $s \in \mathbb{R}$.

For convenience we shall frequently use the short-hand notations

$$S^{\mu}, \quad S_{\text{cl}}^{\mu}, \quad S^{\mu}(\Lambda), \quad S_{\text{cl}}^{\mu}(\Lambda);$$

in particular, the numbers N_0 and N_1 will be indicated only if necessary.

Occasionally we shall also make use of a version of the symbol class S^{μ} where the symbols do not take values in a space of matrices but in a *Fréchet space* E ; we shall denote this space by $S^{\mu}(E)$. A function $a : \mathbb{R}^n \times \mathbb{R}^n \rightarrow E$ belongs to $S^{\mu}(E)$ if it is smooth and satisfies uniform estimates

$$p(D_{\xi}^{\alpha} D_x^{\beta} a(x, \xi)) \leq C_{\alpha\beta p}(a) \langle \xi \rangle^{\mu - |\alpha|}$$

for any continuous semi-norm p on E and any order of derivatives. For example, we can consider $a(x, \xi) \in S^{\mu}(\mathbb{R}^n, \mathbb{R}^n; N_0, N_1)$ as a symbol $a(\xi) \in S^{\mu}(E)$ with $E = \mathcal{C}_b^{\infty}(\mathbb{R}_x^n, \mathbb{C}^{N_1 \times N_0})$, the space of all smooth $\mathbb{C}^{N_1 \times N_0}$ -valued functions having bounded derivatives of any order.

2.2. Pseudodifferential operators on closed manifolds. We let M denote a smooth closed Riemannian manifold of dimension $\dim M = n$. Using a partition of unity and local coordinates and local bundle trivialisations one can define the spaces

$$L_{(\text{cl})}^\mu(M, F_0, F_1), \quad L_{(\text{cl})}^\mu(\Lambda; M, F_0, F_1)$$

of (parameter-dependent) pseudodifferential operators acting between sections into the hermitian vector-bundles F_0 and F_1 . The local operators have symbols as described in the previous subsection with $N_j = \dim F_j$ for $j = 0, 1$, while the spaces of global smoothing operators

$$L^{-\infty}(M, F_0, F_1), \quad L^{-\infty}(\Lambda; M, F_0, F_1)$$

consist of those integral operators on M having an integral kernel belonging to $\mathcal{C}^\infty(F_1 \boxtimes F_0)$, depending continuously on θ and rapidly decreasing on τ in the case of parameter-dependence.

The local homogeneous principal symbols extend to globally defined bundle homomorphisms $\pi^* F_0 \rightarrow \pi^* F_1$, where π denotes the canonical projection $T^*M \setminus \{0\} \rightarrow M$ in case of operators without parameter, otherwise the projection $(T^*M \times \Lambda) \setminus \{0\} \rightarrow M$.

3. Abstract parameter-dependent pseudodifferential operators

In this section we recall and summarize the concept of abstract pseudodifferential operators of Toeplitz type as described in [Se11]. We shall use a slightly modified and simplified notation.

3.1. Abstract pseudodifferential calculi. Let G denote a “set of admissible weights g ” and set $\mathfrak{G} = G \times G$. With any $g \in G$ there is associated a Hilbert space $H(g)$. With any pair $\mathfrak{g} = (g_0, g_1) \in \mathfrak{G}$ we associate a vector space $L^0(\Lambda; \mathfrak{g})$ of “zero-order parameter-dependent operators” and a subspace $L^{-\infty}(\Lambda; \mathfrak{g})$ of “smoothing parameter-dependent operators”, where

$$L^0(\Lambda; \mathfrak{g}) \subset \mathcal{C}(\Lambda, \mathcal{L}(H(g_0), H(g_1))),$$

and the smoothing operators, additionally, are assumed to vanish as $|\lambda| \rightarrow +\infty$. To emphasize the presence of the parameter we shall use notations $A(\lambda)$, $B(\lambda)$, etc. for the elements of $L^0(\Lambda; \mathfrak{g})$.

If $\mathfrak{g}_0 = (g_0, g_1)$ and $\mathfrak{g}_1 = (g_1, g_2)$ are two pairs of admissible weights and $\mathfrak{g}_1 \circ \mathfrak{g}_0 := (g_0, g_2)$, then the (λ -wise) composition of operators is assumed to induce maps

$$L^{\mu_1}(\Lambda; \mathfrak{g}_1) \times L^{\mu_0}(\Lambda; \mathfrak{g}_0) \longrightarrow L^{\mu_1 + \mu_0}(\Lambda; \mathfrak{g}_1 \circ \mathfrak{g}_0),$$

for any choice of $\mu_0, \mu_1 \in \{0, -\infty\}$. If $\mathfrak{g} = (g_0, g_1)$ and $\mathfrak{g}^{(-1)} := (g_0, g_1)$ then taking (λ -wise) the adjoint of operators is supposed to yield mappings

$$L^\mu(\Lambda; \mathfrak{g}) \longrightarrow L^\mu(\Lambda; \mathfrak{g}^{(-1)}).$$

Due to the vanishing at infinity of the smoothing operators, $1 - R(\lambda)$ is invertible for sufficiently large $|\lambda|$ whenever $R(\lambda) \in L^{-\infty}(\Lambda; \mathfrak{g})$ with $\mathfrak{g} = (g, g)$. We shall assume that the inverse again has the same structure, i.e., there exists an $S(\lambda) \in L^{-\infty}(\Lambda; \mathfrak{g})$ such that

$$(1 - R(\lambda))(1 - S(\lambda)) = (1 - S(\lambda))(1 - R(\lambda)) = 1$$

for $|\lambda|$ sufficiently large. This is equivalent to asking that there exists an $R'(\lambda) \in L^{-\infty}(\Lambda; \mathfrak{g})$ such that, for large $|\lambda|$,

$$R'(\lambda) = R(\lambda)(1 - R(\lambda))^{-1}R(\lambda).$$

We shall assume that there exists a “principal symbol”, i.e., a map

$$A(\lambda) \mapsto \sigma(A) = (\sigma_1(A), \dots, \sigma_n(A))$$

assigning to each $A(\lambda) \in L^0(\Lambda; \mathfrak{g})$, $\mathfrak{g} = (g_0, g_1) \in \mathfrak{G}$, an n -tuple of (continuous) bundle homomorphisms

$$(3.1) \quad \sigma_k(A) : E_k(g_0) \longrightarrow E_k(g_1), \quad k = 1, \dots, n,$$

where $E_k(g)$ denotes a Hilbert space bundle¹ associated with the weight $g \in G$. The principal symbol is supposed to vanish on smoothing operators and to be compatible with addition, composition, and taking the adjoint. Moreover, the following are supposed to be equivalent:

- (L1) $A(\lambda) \in L^0(\Lambda; \mathfrak{g})$ is *parameter-elliptic*, i.e., all maps (3.1) are isomorphisms.
- (L2) $A(\lambda) \in L^0(\Lambda; \mathfrak{g})$ has a parametrix $B(\lambda) \in L^0(\Lambda; \mathfrak{g}^{(-1)})$, i.e., both $1 - A(\lambda)B(\lambda)$ and $1 - B(\lambda)A(\lambda)$ are smoothing.

Due to the above described assumption on smoothing operators, for a parameter-elliptic $A(\lambda)$ we can find a parametrix which coincides with $A(\lambda)^{-1}$ for sufficiently large $|\lambda|$.

EXAMPLE 3.1. *Let M be a smooth closed Riemannian manifold. Let the set G of admissible weights consist of all pairs $g = (M, F)$, where F is a smooth hermitian vector bundle over M . For $g = (M, F)$ we define*

$$H(g) := L^2(M, F), \quad E(g) = E_1(g) := \pi^*F,$$

where $\pi : (T^*M \times \Lambda) \setminus \{0\} \rightarrow M$ is the canonical projection. If $\mathfrak{g} = (g_0, g_1)$ with $g_j = (M, F_j)$ we let $L^0(\Lambda; \mathfrak{g})$ denote the space of zero order parameter-dependent classical pseudodifferential operators acting from sections into F_0 to sections into F_1 as described in Section 2.2. The principal symbol $\sigma(A) = \sigma_1(A)$ of $A(\lambda) \in L^0(\Lambda; \mathfrak{g})$ is the homogeneous principal symbol.

¹over some base manifold $B_k(g)$, locally modelled with a finite or infinite dimensional separable Hilbert space

3.2. Parameter-dependent operators of Toeplitz type. Let $\mathbf{g} = (g_0, g_1)$ be a pair of admissible weights and $P_j(\lambda) \in L^0(\Lambda; \mathbf{g}_j)$, $\mathbf{g}_j = (g_j, g_j)$, with $j = 0, 1$ be two projections, i.e., $P_j(\lambda)^2 = P_j(\lambda)$. We then set

$$T^\mu(\Lambda; \mathbf{g}, P_0, P_1) = \left\{ A(\lambda) \in L^\mu(\Lambda; \mathbf{g}) \mid A(\lambda)(1 - P_0(\lambda)) = 0, (1 - P_1(\lambda))A(\lambda) = 0 \right\}$$

with $\mu = 0$ or $\mu = -\infty$. Note that this implies $A(\lambda) = P_1(\lambda)A(\lambda)P_0(\lambda)$ whenever $A(\lambda) \in T^\mu(\Lambda; \mathbf{g}, P_0, P_1)$. A parametrix of such an $A(\lambda)$ is any parameter-dependent operator $B(\lambda) \in L^0(\Lambda; \mathbf{g}^{(-1)}, P_1, P_0)$ such that

$$\begin{aligned} P_0(\lambda) - B(\lambda)A(\lambda) &\in T^{-\infty}(\Lambda; \mathbf{g}_0, P_0, P_0) \\ P_1(\lambda) - A(\lambda)B(\lambda) &\in T^{-\infty}(\Lambda; \mathbf{g}_1, P_1, P_1). \end{aligned}$$

If $P(\lambda) \in L^0(\Lambda; \mathbf{g})$, $\mathbf{g} = (g, g)$, is a projection then so is any associated principal symbol $\sigma_k(P)$. Hence

$$E_k(g, P) := \text{range}(\sigma_k(P) : E_k(g) \rightarrow E_k(g))$$

is a subbundle of $E_k(g)$. For $A(\lambda) \in T^0(\Lambda; \mathbf{g}, P_0, P_1)$ we then define the principal symbol

$$\sigma(A; P_0, P_1) = (\sigma_1(A; P_0, P_1), \dots, \sigma_n(A; P_0, P_1))$$

with

$$(3.2) \quad \sigma_k(A; P_0, P_1) = \sigma_k(A) : E_k(g_0, P_0) \longrightarrow E_k(g_1, P_1), \quad k = 1, \dots, n.$$

The following theorem now holds true:

THEOREM 3.2 (Theorem 3.18 of [Se11]). *Under the above assumptions the following two properties are equivalent:*

(T1) $A(\lambda) \in T^0(\Lambda; \mathbf{g}, P_0, P_1)$ is parameter-elliptic, i.e., all maps (3.2) are isomorphisms.

(T2) $A(\lambda) \in T^0(\Lambda; \mathbf{g}, P_0, P_1)$ has a parametrix $B(\lambda) \in T^0(\Lambda; \mathbf{g}^{(-1)}, P_1, P_0)$.

In this case, one can choose a parametrix $B(\lambda)$ in such a way that $B(\lambda)A(\lambda) = P_0(\lambda)$ and $A(\lambda)B(\lambda) = P_1(\lambda)$ for large enough $|\lambda|$.

Let us note that if we do not have equivalence of (L1) and (L2) but only that (L1) implies (L2) then (T1) implies (T2). Clearly Theorem 3.2 implies that

$$A(\lambda) : H_\lambda(g_0, P_0) \longrightarrow H_\lambda(g_1, P_1)$$

is an isomorphism for large $|\lambda|$, where we have used the notation

$$H_\lambda(g, P) = P(\lambda)(H(g)), \quad P(\lambda) \in L^0(\Lambda; (g, g)).$$

Note that $H_\lambda(g, P)$ is a closed subspace of $H(g)$ for any λ .

EXAMPLE 3.3. *We can apply the above construction to the parameter-dependent classical pseudodifferential operators on M as described in Example 3.1. However, in this way we cannot deal in a satisfactory manner with Toeplitz operators of the form $P_1 A(\lambda) P_0$ where at least one of the projections does not depend on the parameter. This is due to the fact that $L_{\text{cl}}^0(M, F, F) \not\subset L_{\text{cl}}^0(\Lambda; M, F, F)$; in fact*

$$L_{\text{cl}}^0(M, F_0, F_1) \cap L_{\text{cl}}^0(\Lambda; M, F_0, F_1) = \text{Hom}(F_0, F_1),$$

the space of bundle homomorphisms $F_0 \rightarrow F_1$. In the next section we develop a calculus avoiding this problem.

Let us also remark that Grubb in [Gru85] introduced a calculus that allows to consider fixed operators as parameter-dependent ones, but that the concept of parameter-ellipticity in this calculus requires positive regularity of the parameter-dependent operators (we do not go into details here, but only mention that the regularity somehow measures the deviation of the parameter-dependence from the one described in Section 2) and fixed operators in general only have regularity 0.

4. A calculus of parameter-dependent operators on \mathbb{R}^n

In \mathbb{R}^n we have a one-to-one correspondence between pseudodifferential operators $A = \text{op}(a)$ and their symbols. For this reason we shall mainly work on the level of symbols. The composition of operators corresponds to the Leibniz product of symbols, defined by the (oscillatory) integral

$$(a \# b)(x, \xi) = \iint e^{-iy\eta} a(x, \xi + \eta) b(x + y, \xi) dy d\eta,$$

while taking the (formal) adjoint of A with respect to the L_2 -scalar product corresponds to passing to the symbol

$$a^{(*)}(x, \xi) = \iint e^{-iy\eta} a(x + y, \xi)^* dy d\eta.$$

Note that both maps $(a, b) \mapsto a \# b : S^{\mu_1}(N_1, N_2) \times S^{\mu_0}(N_0, N_1) \rightarrow S^{\mu_0 + \mu_1}(N_0, N_2)$ and $a \mapsto a^{(*)} : S^{\mu}(N_0, N_1) \rightarrow S^{\mu}(N_1, N_0)$ are continuous.

4.1. A first calculus of parameter-dependent symbols. We begin with a first, "rough" calculus that will be refined later on.

DEFINITION 4.1. *With real μ or $\mu = -\infty$ let $\mathbf{S}^{\mu}(\Lambda) = \mathbf{S}^{\mu}(\Lambda; N_0, N_1)$ denote the space of all continuous and bounded functions $a : \Lambda \rightarrow S^{\mu}$ for whom there exists a continuous function $a^{\infty} : [a, b] \rightarrow S^{\mu}$ such that*

$$(4.1) \quad a(\tau, \cdot) \xrightarrow{\tau \rightarrow \infty} a^{\infty} \quad \text{in } \mathcal{C}([a, b], S^{\mu+1})$$

(note the order $\mu + 1$ in (4.1)). We call a^{∞} the limit-family of a . By $\mathbf{S}_0^{\mu}(\Lambda)$ we denote the subspace of symbols with vanishing limit-family.

Obviously $S^\mu \subset \mathbf{S}^\mu(\Lambda)$. The calculus is closed under composition (Leibniz product) and taking the adjoint. The limit-family behaves multiplicative under composition. Composition with the symbol $\langle \xi \rangle^\nu$ (both from the left or the right) yields isomorphisms $\mathbf{S}^\mu(\Lambda) \rightarrow \mathbf{S}^{\mu+\nu}(\Lambda)$.

Let us remark that in the previous definition it would be equivalent to ask only for $a^\infty \in \mathcal{C}([a, b], S^{\mu+1})$, since it then follows that $a^\infty \in \mathcal{C}([a, b], S^\mu)$. In fact, this is true, since we can identify $\mathcal{C}([a, b], S^\nu)$ with $S^\nu(E)$ for $E = \mathcal{C}([a, b])$ and then use following general observation:

LEMMA 4.2. *Let E be a Fréchet space and let (a_n) be a bounded sequence in $S^\mu(E)$ that converges in $S^{\mu+1}(E)$ to a^∞ . Then $a^\infty \in S^\mu(E)$.*

PROOF. Since $S^\mu(E) \hookrightarrow \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n, E)$ continuously, a^∞ is a smooth function with values in E . Moreover, for any semi-norm p of E we have

$$p(D_\xi^\alpha D_x^\beta a^\infty(x, \xi)) \xleftarrow{n \rightarrow \infty} p(D_\xi^\alpha D_x^\beta a_n(x, \xi)) \leq C \langle \xi \rangle^{\mu-|\alpha|}$$

with a constant C independent of (x, ξ) and n . This yields the claim. \square

The next lemma states that asymptotic summation is possible within the calculus.

LEMMA 4.3. *Given $a_{\mu-k} \in \mathbf{S}^{\mu-k}(\Lambda)$, $k \in \mathbb{N}_0$, there exists an $a \in \mathbf{S}^\mu(\Lambda)$ such that $a - \sum_{k=0}^{\ell-1} a_{\mu-k}$ belongs to $\mathbf{S}^{\mu-\ell}(\Lambda)$ for any ℓ .*

PROOF. Let K be $\mathbb{R}_+ \cup \{\infty\}$ the one-point completion of \mathbb{R}_+ at infinity and

$$E_0 = \mathcal{C}_b(\Lambda), \quad E_1 = \mathcal{C}(K \times [a, b]),$$

where $\mathcal{C}_b(\Lambda)$ is the space of continuous and bounded functions on Λ . With a fixed zero excision function $\chi(\xi)$ we can choose a zero sequence (c_k) such that any series

$$b_\ell(x, \xi, \lambda) := \sum_{k=\ell}^{\infty} \chi(c_k \xi) a_{\mu-k}(x, \xi, \lambda), \quad \ell \in \mathbb{N}_0,$$

converges both in $S^{\mu-\ell}(E_0)$ and $S^{\mu+1-\ell}(E_1)$ for any ℓ . Since taking the limit (i.e., evaluation in $\tau = \infty$) is a continuous map $E_1 \rightarrow E_2 := \mathcal{C}([a, b])$, we see that

$$\lim_{\tau \rightarrow \infty} b_\ell(x, \xi, \tau, \theta) = \sum_{k=\ell}^{\infty} \chi(c_k \xi) a_{\mu-k}^\infty(x, \xi, \theta)$$

with convergence in $S^{\mu-\ell+1}(E_2)$. However we can modify (i.e., diminish) the c_k in such a way that the above convergences remain valid and, additionally, the last series converges in $C([a, b], S^{\mu-\ell})$ for any ℓ . Hence we can take $a = b_0$. \square

EXAMPLE 4.4. *Let $a : \Lambda \rightarrow S^\mu$ be continuous and bounded. Assume that a considered as a function with values in $S^{\mu+1}$ is continuously differentiable with respect to*

τ and that there exists a $\delta > 0$ such that $(1 + \tau)^{1+\delta} \partial_\tau a$ is bounded. Then $a \in \mathbf{S}^\mu(\Lambda)$. In fact, (4.1) holds true if we set

$$a^\infty = a(1) + \int_1^\infty \partial_\tau a(\tau, \cdot) d\tau$$

(observe Lemma 4.2 and the comment given before).

EXAMPLE 4.5. Let $a \in S_{\text{cl}}^0(\Lambda)$ be a classical symbol of order $\mu = 0$. Then a belongs to $\mathbf{S}^0(\Lambda)$ and has limit-family

$$a^\infty(x, \theta) = a^{(0)}(x, 0, 1, \theta) \quad (\text{independent of } \xi),$$

where $a^{(0)}$ is the homogeneous principal symbol of a . In fact, first it is easy to see that analogous symbols of order -1 belong to $\mathbf{S}^0(\Lambda)$ and have vanishing limit-family. Thus we can assume that a is homogeneous in the large, i.e., satisfies

$$a(x, t\xi, t\tau, \theta) = a(x, \xi, \tau, \theta) \quad \forall t \geq 1 \quad \forall |(\xi, \tau)| \geq 1.$$

Writing $a(x, \xi, \tau, \theta) = a(x, \xi/\tau, 1, \theta)$ for $\tau \geq 1$, we see that the limit-family must be as stated if it exists. Moreover,

$$\tau^2 \partial_\tau a(x, \xi, \tau, \theta) = - \sum_{j=1}^n \xi_j (\partial_{\xi_j} a)(x, \xi/\tau, 1, \theta) = - \sum_{j=1}^n \xi_j \tau (\partial_{\xi_j} a)(x, \xi, \tau, \theta).$$

Using that $\partial_{\xi_j} a \in S_{\text{cl}}^{-1}(\Lambda)$ it follows easily that $\tau^2 \partial_\tau a$ is bounded with values in S^1 and we can apply the previous example.

4.2. Parameter-ellipticity and invertibility for large parameters. Let us now discuss the concept of ellipticity for the classes $\mathbf{S}^\mu(\Lambda)$. Throughout this subsection we assume $N_0 = N_1 = N$ for some $N \in \mathbb{N}$. Let us first observe the following:

PROPOSITION 4.6. Let $r \in \mathbf{S}_0^{-\infty}(\Lambda)$. Then there exists an $s \in \mathbf{S}_0^{-\infty}(\Lambda)$ such that

$$(1 + r(\lambda)) \# (1 + s(\lambda)) = (1 + s(\lambda)) \# (1 + r(\lambda)) = 1$$

provided $|\lambda|$ is large enough. In particular, $1 + r(\lambda)$ is invertible in S^0 for large $|\lambda|$.

PROOF. Let us recall that set of invertible elements in S^0 form an open set and that inversion is a continuous map². Since $r(\lambda) \rightarrow 0$ in S^0 for $|\lambda| \rightarrow \infty$, we can conclude the existence of a C such that $(1 - r(\lambda))^{-1}$ exists in S^0 for $|\lambda| \geq C$ and is continuous and bounded as a function of λ . Whenever the inverse exists,

$$(1 + r(\lambda))^{-1} = 1 - r(\lambda) + r(\lambda) \# (1 + r(\lambda))^{-1} \# r(\lambda).$$

Thus if $\chi(t)$ is a zero-excision function vanishing for $t \leq C$ then

$$s(\lambda) = -r(\lambda) + \chi(|\lambda|) r(\lambda) \# (1 + r(\lambda))^{-1} \# r(\lambda)$$

is the desired element of $\mathbf{S}_0^{-\infty}(\Lambda)$. □

²In fact, S^0 is a Ψ^* -algebra in the sense of Gramsch [Gra84], and continuity of inversion is a consequence of a result due to Waelbroeck [Wa71].

We shall call $a \in \mathbf{S}^\mu(\Lambda)$ *parameter-elliptic* provided the following two conditions hold:

- (I) There exists a $C \geq 0$ such that $a(x, \xi, \lambda)$ is invertible whenever $|\xi| \geq C$ and $|a(x, \xi, \lambda)^{-1}| \langle \xi \rangle^\mu$ is uniformly bounded in $x \in \mathbb{R}^n$, $\lambda \in \Lambda$, and $|\xi| \geq C$.
- (II) The limit family a^∞ is invertible in S^μ (pointwise for each $\theta \in [a, b]$).

Note that in (II) we could equivalently ask that a^∞ is pointwise invertible as a map $H^s(\mathbb{R}^n) \rightarrow H^{s-\mu}(\mathbb{R}^n)$ for some $s \in \mathbb{R}$.

PROPOSITION 4.7. *Let $a \in \mathbf{S}^\mu(\Lambda)$ be parameter-elliptic as described above. Then we can choose a zero excision function $\chi(\xi)$ vanishing for $|\xi| \leq C$ such that $b(x, \xi, \lambda) := \chi(\xi)a(x, \xi, \lambda)^{-1}$ belongs to $\mathbf{S}^{-\mu}(\Lambda)$ and both $a(\lambda)\#b(\lambda) - 1$ and $b(\lambda)\#a(\lambda) - 1$ belong to $\mathbf{S}^{-1}(\Lambda)$.*

PROOF. Without loss of generality $\mu = 0$. Let $a^\infty \in S^0$ be the limit-family. The continuity of inversion in S^0 yields the existence of a $D \geq 0$ such that $a^\infty(x, \xi, \theta)$ is invertible for $|\xi| \geq D$ with inverse uniformly bounded in (x, ξ, θ) . By enlarging one constant or the other we may assume $C = D$. With $b^\infty(x, \xi, \theta) = \chi(\xi)a^\infty(x, \xi, \theta)^{-1}$ we have

$$\begin{aligned} b(x, \xi, \tau, \theta) - b^\infty(x, \xi, \theta) \\ = \chi(\xi)a(x, \xi, \tau, \theta)^{-1}(a^\infty(x, \xi, \theta) - a(x, \xi, \tau, \theta))a^\infty(x, \xi, \theta)^{-1}. \end{aligned}$$

This implies that $b(\tau, \cdot) \xrightarrow{\tau \rightarrow \infty} b^\infty$ in $\mathcal{C}([a, b], S^1)$. Thus $b \in \mathbf{S}^0(\Lambda)$. The remaining claim follows by using the explicit formula for the remainder term $r := a\#b - ab$, namely

$$r(x, \xi, \lambda) = \sum_{|\alpha|=1} \iint e^{-iy\eta} D_\xi^\alpha a(x, \xi + \eta, \lambda) \partial_x^\alpha b(x + y, \xi, \lambda) dy d\eta,$$

and analogously for $b\#a - ba$. \square

THEOREM 4.8. *Let $a \in \mathbf{S}^\mu(\Lambda)$ be parameter-elliptic. Then there exists a $b \in \mathbf{S}^{-\mu}(\Lambda)$ such that both $a(\lambda)\#b(\lambda) - 1$ and $b(\lambda)\#a(\lambda) - 1$ belong to $\mathbf{S}_0^{-\infty}(\Lambda)$ and vanish for large enough $|\lambda|$. In particular, $a(\lambda)$ is invertible in S^0 for sufficiently large $|\lambda|$ with $a(\lambda)^{-1} = b(\lambda)$.*

PROOF. Using the previous proposition and the usual von Neumann argument together with Lemma 4.3 we can construct a parametrix b' modulo $\mathbf{S}^{-\infty}(\Lambda)$. With $r(\lambda) = 1 - a(\lambda)\#b'(\lambda)$ we then have $r^\infty = 1 - a^\infty\#b'^\infty$. Hence, using (II),

$$b''(\tau, \theta) := b'(\tau, \theta) + a^\infty(\theta)^{-1}r^\infty(\theta)$$

is a parametrix modulo $\mathbf{S}_0^{-\infty}(\Lambda)$. In fact, $b'' - b' \in \mathbf{S}^{-\infty}(\Lambda)$, since $r^\infty \in \mathcal{C}([a, b], S^{-\infty})$, and

$$(1 - a\#b'')^\infty = (1 - a\#b')^\infty - a^\infty(a^\infty)^{-1}r^\infty = r^\infty - r^\infty = 0.$$

It remains to apply Proposition 4.6 to finally obtain the desired parametrix. \square

EXAMPLE 4.9. Let $a \in S_{\text{cl}}^0(\Lambda)$ be a classical symbol of order 0. Then parameter-ellipticity in the sense of (I) and (II) is equivalent to the usual parameter-ellipticity of classical symbols, i.e., the principal symbol $a^{(0)}$ is everywhere invertible and there exists a C such that

$$|a^{(0)}(x, \xi, \lambda)^{-1}| \leq C \quad x \in \mathbb{R}^n, \quad |(\xi, \lambda)| = 1.$$

Note that the latter estimate is only a condition for $|x| \rightarrow \infty$ and can be omitted for example when $a^{(0)}$ is constant in x for large $|x|$.

The calculus described above is complete but still not suited for our purposes, since the ellipticity condition (I) not only asks the invertibility of a certain principal symbol but also requires an estimate on the inverted symbol. Our next aim is to single out a subcalculus in which (I) can be replaced by a condition avoiding such kind of estimates.

4.3. A class of homogeneous symbols. We describe here a class of homogeneous (in the large) symbols not only containing $S_{\text{cl}}^0(\Lambda)$ as a subclass but also homogeneous symbols which are constant in the parameter λ . For convenience of notation we now consider symbols independent of the x -variable and assume that $\Lambda = \overline{\mathbb{R}}_+$ (so there is no angular variable θ); the general case is completely analogous, cf. Remark 4.12 below. Let

$$\mathbb{S}_+^n = \{(\xi, \tau) \mid \xi \in \mathbb{R}^n, \tau \geq 0, |(\xi, \tau)| = 1\}$$

be the closed upper semi-sphere in \mathbb{R}^{n+1} and, with $\mathbf{n} := (0, 1)$ the “north-pole”,

$$(4.2) \quad \widehat{\mathbb{S}}_+^n = \mathbb{S}_+^n \setminus \mathbf{n}.$$

On $\widehat{\mathbb{S}}_+^n$ we shall make use of polar-coordinates

$$\xi = \sin \rho \cdot \phi, \quad \tau = \cos \rho \quad (\phi \in \mathbb{S}^{n-1}, 0 < \rho \leq \pi/2),$$

respectively $\phi = \xi/|\xi|$ and $\rho = \arccos \tau$. Moreover, we let $\mathcal{C}^{\infty, \gamma}(\widehat{\mathbb{S}}_+^n)$, $\gamma \in \mathbb{R}$, denote the space of all smooth functions \widehat{a} on $\widehat{\mathbb{S}}_+^n$ having the property that

$$\rho^{-\gamma+\varepsilon}(\rho \partial_\rho)^j \Delta_\phi^k \widehat{a} \in L^\infty(\widehat{\mathbb{S}}_+^n) \quad \forall j, k \in \mathbb{N}_0 \quad \forall \varepsilon > 0.$$

DEFINITION 4.10. We say that $\widehat{a} \in \mathcal{C}^\infty(\widehat{\mathbb{S}}_+^n)$ has (generalized) Taylor asymptotics at the north-pole \mathbf{n} if

$$(4.3) \quad \widehat{a}(\phi, \rho) \sim \sum_{j=0}^{\infty} \rho^j \widehat{a}_j(\phi), \quad \widehat{a}_j \in \mathcal{C}^\infty(\mathbb{S}^{n-1}),$$

for suitable \widehat{a}_j , where \sim means that, for any $\ell \in \mathbb{N}$,

$$(4.4) \quad \widehat{a}(\phi, \rho) - \sum_{j=0}^{\ell} \rho^j \widehat{a}_j(\phi) \in \mathcal{C}^{\infty, \ell+1}(\widehat{\mathbb{S}}_+^n).$$

The space consisting of all such functions we denote by $\mathcal{C}_T^\infty(\widehat{\mathbb{S}}_+^n)$.

REMARK 4.11. If $\hat{a} \in \mathcal{C}_T^\infty(\widehat{\mathbb{S}}_+^n)$ is as in (4.3) then

$$\hat{a}_0(\phi) = \lim_{\rho \rightarrow 0} \hat{a}(\sin \rho \cdot \phi, \cos \rho).$$

In particular, if $\hat{a} \in \mathcal{C}^\infty(\mathbb{S}_+^n)$ is smooth on the whole upper semi-sphere then

$$\hat{a}_0(\phi) = \hat{a}(0, 1) = \hat{a}(\mathbf{n})$$

is constant in ϕ and coincides with the value of \hat{a} in the north-pole \mathbf{n} .

REMARK 4.12. The definitions of $\mathcal{C}^{\infty, \gamma}(\widehat{\mathbb{S}}_+^n)$ and $\mathcal{C}_T^\infty(\widehat{\mathbb{S}}_+^n)$ can be easily generalized to the E -valued setting for a Fréchet space E , resulting in spaces $\mathcal{C}^{\infty, \gamma}(\widehat{\mathbb{S}}_+^n, E)$ and $\mathcal{C}_T^\infty(\widehat{\mathbb{S}}_+^n, E)$, respectively. Choosing $E = \mathcal{C}([a, b]_\theta, \mathcal{C}_b^\infty(\mathbb{R}_x^n))$ we obtain symbols with variable x -coefficients and defined for a general strip Λ .

It is straightforward to see that $\mathcal{C}_T^\infty(\widehat{\mathbb{S}}_+^n)$ is closed under taking inverses in the following sense:

LEMMA 4.13. $\hat{a} \in \mathcal{C}_T^\infty(\widehat{\mathbb{S}}_+^n)$ has an inverse $\hat{a}^{-1} \in \mathcal{C}_T^\infty(\widehat{\mathbb{S}}_+^n)$ if, and only if, both \hat{a} and \hat{a}_0 are pointwise invertible.

Now let $\hat{a} \in \mathcal{C}_T^\infty(\widehat{\mathbb{S}}_+^n)$ and $\chi(\xi)$ be a zero-excision function. Consider

$$(4.5) \quad a(\xi, \tau) = \chi(\xi) \hat{a}\left(\frac{(\xi, \tau)}{|(\xi, \tau)|}\right), \quad (\xi, \tau) \in \mathbb{R}^n \times \overline{\mathbb{R}}_+.$$

PROPOSITION 4.14. $a(\xi, \tau)$ as in (4.5) is a smooth function satisfying, for any $\alpha \in \mathbb{N}_0^n$ and $k \in \mathbb{N}_0$, the uniform estimates

$$(4.6) \quad |D_\xi^\alpha D_\tau^k a(\xi, \tau)| \leq C_{\alpha k} \langle \xi \rangle^{-|\alpha|} \langle \xi, \tau \rangle^{-k}.$$

PROOF. (i) Consider the case $\hat{a}(\phi, \rho) = \rho$, i.e.,

$$a(\xi, \tau) = \chi(\xi) \arccos \frac{\tau}{|(\xi, \tau)|}.$$

Choose a zero-excision function $\chi_1(\xi, \tau)$ such that $\chi_1 \chi = \chi$. Then

$$\begin{aligned} \partial_\tau a(\xi, \tau) &= -\left(1 - \frac{\tau}{|(\xi, \tau)|}\right)^{-1/2} \left(|(\xi, \tau)|^{-1} - \tau^2 |(\xi, \tau)|^{-3}\right) \chi(\xi) \\ &= \chi(\xi) |\xi| \cdot \chi_1(\xi, \tau) |(\xi, \tau)|^{-2}. \end{aligned}$$

The first factor is a symbol in S^1 , while the second is a symbol in $S_{\text{cl}}^{-2}(\overline{\mathbb{R}}_+)$. This yields (4.6) in case $k \geq 1$. Moreover,

$$(4.7) \quad \partial_{\xi_i} \arccos \frac{\tau}{|(\xi, \tau)|} = \frac{\xi_i}{|\xi|} \cdot \frac{\tau}{\tau^2 + |\xi|^2}$$

is a product of a function positively homogeneous of degree 0 in ξ and one of degree -1 in (ξ, τ) . We obtain (4.6) in case $k = 0$. By Leibniz rule then (4.6) also holds for $\hat{a}(\phi, \rho) = \rho^j$ with $j \in \mathbb{N}$.

(ii) If $\hat{a}(\phi, \rho) = \hat{a}(\phi)$ then $a(\xi, \tau) = \chi(\xi) \hat{a}(\xi/|\xi|)$ is a symbol in S^0 and (4.6) is valid.

(iii) Let $\widehat{r} \in \mathcal{C}^{\infty, \ell+1}(\widehat{\mathbb{S}}_+^n)$ and A be a differential operator of order k on \mathbb{S}_+^n with smooth coefficients. In polar-coordinates A takes the form

$$\rho^{-k} \sum_{i=0}^k A_i(\rho) (\rho \partial_\rho)^i, \quad A_i(\rho) \in \mathcal{C}^\infty([0, \pi/2], \text{Diff}^{k-i}(\mathbb{S}^{n-1})).$$

Therefore $A\widehat{r} = O(\rho^\varepsilon)$ for any A of order $k < \ell + 1$. We conclude that $\widehat{r} \in \mathcal{C}^\ell(\mathbb{S}_+^n)$. It follows that $\chi(\xi)\widehat{r}((\xi, \tau)/|(\xi, \tau)|)$ satisfies (4.6) provided $|\alpha| + k \leq \ell$.

(iv) To complete the proof it remains to combine (i)-(iii) with the expansion (4.4) which is valid for arbitrary ℓ . \square

THEOREM 4.15. *Let $a(\xi, \tau)$ be as in (4.5) with \widehat{a} as in (4.3). Then $a \in \mathbf{S}^0(\Lambda)$ with limit-family³ $a^\infty(\xi) = \chi(\xi)\widehat{a}_0(\xi/|\xi|)$.*

PROOF. Due to Proposition 4.14 we only need to verify the existence of the limit-family. Obviously we may assume that $\widehat{a}_0 = 0$ and then show that the limit-family is zero. Since $\widehat{a}_0 = 0$ we have $\widehat{a} = \rho\widehat{b}$ with $\widehat{b} \in \mathcal{C}_T^\infty(\widehat{\mathbb{S}}_+^n)$. If we associate b with \widehat{b} then $b \in \mathbf{S}^0(\Lambda)$. Therefore it suffices to assume $\widehat{a} = \rho$ and show that the limit-family exists and equals zero. In this case observe that

$$a(\xi, \tau) = \chi(\xi)\widehat{a}\left(\frac{(\xi, \tau)}{|(\xi, \tau)|}\right) = \chi(\xi) \arccos \frac{\tau}{|(\xi, \tau)|}$$

converges to zero uniformly on compact subsets of \mathbb{R}^n as τ tends to $+\infty$. Since \arccos is bounded it follows that, for $\alpha = 0$,

$$(4.8) \quad \sup_{\xi \in \mathbb{R}} |D_\xi^\alpha a(\xi, \tau)| \langle \xi \rangle^{|\alpha|-1} \xrightarrow{\tau \rightarrow +\infty} 0.$$

It remains to verify (4.8) for arbitrary $\alpha \neq 0$. However, this follows easily from the case $\alpha = 0$ and product rule. \square

4.4. The refined calculus. With $\mathbf{S}^{(0)}(\Lambda)$ we shall denote the space of all symbols of the form

$$(4.9) \quad a(x, \xi, \tau, \theta) = \widehat{a}\left(x, \frac{(\xi, \tau)}{|(\xi, \tau)|}, \theta\right) \quad (\text{with } \xi \neq 0)$$

with $\widehat{a} \in \mathcal{C}_T^\infty(\widehat{\mathbb{S}}_+^n, E)$ with $E = \mathcal{C}([a, b]_\theta, \mathcal{C}_b^\infty(\mathbb{R}_x^n))$, cf. Remark 4.12. Moreover, we assume that \widehat{a} is constant in x for large $|x|$ (this assumption is devoted to the non-compactness of \mathbb{R}^n).

DEFINITION 4.16. *We define $\mathbf{S}_{\text{w-cl}}^0(\Lambda)$ as the subspace of $\mathbf{S}^0(\Lambda)$ consisting of all symbols a for which there exists an $a^{(0)} \in S^{(0)}(\Lambda)$ such that*

$$a - \chi(\xi)a^{(0)} \in S^{-1}(\Lambda)$$

³if $\Lambda = \overline{\mathbb{R}}_+$ the family is of course independent of θ , hence a fixed operator

for some (and then for any) zero-excision function χ . We call $a^{(0)}$ the homogeneous principal symbol of a . The associated symbol $\widehat{a}_0^{(0)}$, the leading term in (4.3), we shall refer to as the angular symbol of a . Recall from Remark 4.11 that

$$(4.10) \quad \widehat{a}_0^{(0)}(x, \xi, \theta) = \lim_{\rho \rightarrow 0} a^{(0)}(x, \sin \rho \cdot \xi, \cos \rho, \theta) \quad (|\xi| = 1).^4$$

The subscript w-cl stands for "weakly classical". The homogeneous principal symbol is well-defined (i.e., uniquely determined by a) and thus is the angular symbol. We shall call $a \in \mathbf{S}_{\text{w-cl}}^0(\Lambda)$ *parameter-elliptic* provided the following two conditions hold:

- (S1) Both the homogeneous principal symbol $a^{(0)}$ and the angular symbol $\widehat{a}_0^{(0)}$ are (pointwise) invertible.
- (S2) The limit family a^∞ is pointwise invertible in S^0 .

Recall that (S1) is equivalent to the invertibility of $a^{(0)}$ in the class $\mathbf{S}^{(0)}(\Lambda)$. It is then clear that (S1) implies the ellipticity-condition (I) of Section 4.2 (with $\mu = 0$) and we obtain a parametrix that coincides with the inverse for large values of the parameter:

THEOREM 4.17. *Let $a \in \mathbf{S}_{\text{w-cl}}^0(\Lambda)$ be parameter-elliptic. Then there exists a $b \in \mathbf{S}_{\text{w-cl}}^0(\Lambda)$ such that both $a(\lambda) \# b(\lambda) - 1$ and $b(\lambda) \# a(\lambda) - 1$ belong to $S_0^{-\infty}(\Lambda)$ and vanish for large enough $|\lambda|$.*

4.5. Weakly classical operators. In the previous sections we worked exclusively on the level of symbols. We now pass to the corresponding spaces of pseudodifferential operators and write

$$\mathbf{L}_{\text{w-cl}}^0(\Lambda; \mathbb{R}^n, N_0, N_1) = \left\{ A(\lambda) = \text{op}(a)(\lambda) \mid a \in S_{\text{w-cl}}^0(\Lambda; N_0, N_1) \right\}.$$

Similarly we obtain the space $\mathbf{L}_0^{-\infty}(\Lambda; \mathbb{R}^n, N_0, N_1)$ using smoothing symbols.

With $A(\lambda) = \text{op}(a)(\lambda) \in \mathbf{L}_{\text{w-cl}}^0(\Lambda; \mathbb{R}^n, N_0, N_1)$ we associate a principal symbol $\sigma(A)$ which has three components: The first component, $\sigma_1(A)$, is the homogeneous principal symbol of a , considered as a bundle morphism⁵

$$(\mathbb{R}_x^n \times (\mathbb{R}_\xi^n \setminus \{0\}) \times \Lambda) \times \mathbb{C}^{N_0} \longrightarrow (\mathbb{R}_x^n \times (\mathbb{R}_\xi^n \setminus \{0\}) \times \Lambda) \times \mathbb{C}^{N_1}.$$

The second component, $\sigma_2(A)$, is the angular symbol of a , considered as a bundle morphism

$$(\mathbb{R}_x^n \times \mathbb{S}_\xi^{n-1} \times [a, b]) \times \mathbb{C}^{N_0} \longrightarrow (\mathbb{R}_x^n \times \mathbb{S}_\xi^{n-1} \times [a, b]) \times \mathbb{C}^{N_1}.$$

The third and last component, $\sigma_3(A)$, is the operator-family associated with the limit-family of a , considered as a bundle morphism

$$[a, b] \times L^2(\mathbb{R}^n) \longrightarrow [a, b] \times L^2(\mathbb{R}^n).$$

⁴Note that with $a^{(0)}$ also $\widehat{a}_0^{(0)}$ is independent of x for large $|x|$

⁵by considering an $(N_1 \times N_1)$ -matrix valued function on some space X as a bundle morphism between the trivial bundles $X \times \mathbb{C}^{N_0}$ and $X \times \mathbb{C}^{N_1}$

We can now apply in this setting the abstract approach described in Section 3 and obtain resulting classes of Toeplitz operators

$$\mathbf{L}_{\text{w-cl}}^0(\Lambda; \mathbb{R}^n, (N_0, P_0), (N_1, P_1)) = P_1(\lambda) \mathbf{L}_{\text{w-cl}}^0(\Lambda; \mathbb{R}^n, N_0, N_1) P_0(\lambda)$$

with the corresponding definition of the principal symbol.

4.6. Weakly classical operators on smooth manifolds. We finally indicate how to modify the above constructions to deal with operators on a smooth closed Riemannian manifold M of dimension n . With hermitean vector bundles F_0 and F_1 over M we define $\mathbf{L}^\mu(\Lambda; M, F_0, F_1)$ as the space of all continuous and bounded functions $A : \Lambda \rightarrow L^\mu(M, F_0, F_1)$ that have a limit-family

$$A^\infty \in \mathcal{C}([a, b], L^\mu(M, F_0, F_1)),$$

analogously defined as in Definition 4.1, replacing S^μ by $L^\mu(M, F_0, F_1)$. Those A having vanishing limit-family form the subspace $\mathbf{L}_0^\mu(\Lambda; M, E_0, E_1)$. Moreover, $\mathbf{L}_{\text{w-cl}}^0(\Lambda; M, F_0, F_1)$ is the subspace of those operators having in local coordinates a weakly classical symbol in the sense of Definition 4.16. The local homogeneous principal symbols of $A(\lambda)$ globally define a homogeneous principal symbol

$$(4.11) \quad \sigma_1(A) : \pi^* F_0 \longrightarrow \pi^* F_1, \quad \pi : (T^*M \setminus \{0\}) \times \Lambda \rightarrow M,$$

where π denotes the canonical projection onto M . Furthermore, the local angular symbols of $A(\lambda)$ globally yield a morphism

$$(4.12) \quad \sigma_2(A) : \pi_1^* F_0 \longrightarrow \pi_1^* F_1, \quad \pi_1 : S^*M \rightarrow M,$$

where π_1 is the canonical projection of the co-sphere bundle onto M . The limit-family we consider as a morphism

$$(4.13) \quad \sigma_3(A) : [a, b] \times L^2(M, F_0) \longrightarrow [a, b] \times L^2(M, F_1)$$

between trivial bundles. Parameter-ellipticity of $A(\lambda)$ means bijectivity of all three morphisms $\sigma_j(A)$. Again we can apply the abstract approach of Section 3 to obtain resulting classes of Toeplitz operators

$$(4.14) \quad \mathbf{L}_{\text{w-cl}}^0(\Lambda; M, (E_0, P_0), (E_1, P_1)) = P_1(\lambda) \mathbf{L}_{\text{w-cl}}^0(\Lambda; M, E_0, E_1) P_0(\lambda)$$

where $P_j(\lambda) \in \mathbf{L}_{\text{w-cl}}^0(\Lambda; M, E_j, E_j)$ for $j = 0, 1$, including the corresponding notion of principal symbol and parameter-ellipticity.

5. Applications

We shall discuss two applications. For both we shall need the following lemma:

LEMMA 5.1. *Let $p \in S^\mu(N_0, N_1)$, $r_j(\lambda) \in S_{\text{cl}}^{\mu_j}(\Lambda; N_j, N_j)$, and $a(\lambda) = r_1(\lambda) \# p \# r_0(\lambda)$. Then:*

a) For any order of derivatives, $a(\lambda)$ satisfies uniform estimates

$$|D_\xi^\alpha D_x^\beta D_\tau^k a(x, \xi, \tau, \theta)| \leq C_{\alpha\beta k} \langle \xi \rangle^{\mu - |\alpha|} \langle \xi, \tau \rangle^{\mu_0 + \mu_1 - k}.$$

b) If $\mu_0 = -\mu_1$ then $a(\lambda)$ belongs to $\mathbf{S}^\mu(\Lambda; N_0, N_1)$ and has limit-family

$$a^\infty(\theta) = r_0^\infty(\theta) \# p \# r_1^\infty(\theta),$$

where $r_j^\infty(x, \theta) = r_j^{(\mu_j)}(x, 0, 1, \theta)$ with the homogeneous principal symbol $r_j^{(\mu_j)}(x, \xi, \tau, \theta)$ of r_j .

PROOF. Part a) is standard. Part b) is a version of Example 4.5. In fact, first we may assume that r_j is homogeneous in the large of order μ_j , since the lower order terms due to a) yield elements in $\mathbf{S}^\mu(\Lambda)$ with vanishing limit-family. Then we can write, for $\tau \geq 1$,

$$a(x, \xi, \tau, \theta) = r_1(x, \xi/\tau, 1, \theta) \# p(x, \xi) \# r_0(x, \xi/\tau, 1, \theta)$$

and derive similarly as in Example 4.5 and using a) that $\tau^2 \partial_\tau a$ is bounded with values in $S^{\mu+1}$. It remains to apply Example 4.4. \square

5.1. Classical operators in projected subspaces. Let $A(\lambda) \in L_{\text{cl}}^\mu(\Lambda; M, F_0, F_1)$ be a classical pseudodifferential operator of order $\mu \in \mathbb{R}$ and $P_j \in L_{\text{cl}}^0(M, F_j, F_j)$, $j = 0, 1$, be two pseudodifferential projections not depending on the parameter λ . Setting

$$H^s(M, F_j, P_j) = P_j(H^s(M, F_j)), \quad s \in \mathbb{R},$$

we want to derive a criterion ensuring the invertibility of

$$(5.1) \quad P_1 A(\lambda) P_0 : H^s(M, F_0, P_0) \longrightarrow H^{s-\mu}(M, F_1, P_1)$$

for sufficiently large $|\lambda|$. In fact, this will be a consequence of Theorem 5.2, below.

Let us denote by $a^{(\mu)}$ and $p_j^{(0)}$ the homogeneous principal symbol of $A(\lambda)$ and P_j , respectively, and, with π and π_1 as in (4.11) and (4.12), write

$$E^0(P_j) := p_j^{(0)}(\pi^* F_j) \subset \pi^* F_j, \quad E^1(P_j) := p_j^{(0)}(\pi_1^* F_j) \subset \pi_1^* F_j.$$

THEOREM 5.2. *With the previously introduced notation assume that the following mappings are isomorphisms:*

- (1) $p_1^{(0)} a^{(\mu)} p_0^{(0)} : E^0(P_0) \rightarrow E^0(P_1)$,
- (2) $p_1^{(0)} a^{(\mu)}|_{(\xi, \lambda)=(0,1)} p_0^{(0)} : E^1(P_0) \rightarrow E^1(P_1)$,
- (3) $P_1 a^{(\mu)}|_{(\xi, \lambda)=(0,1)} P_0 : H^0(M, E_0, P_0) \rightarrow H^0(M, E_1, P_1)$

(in (2) we consider $a^{(\mu)}|_{(\xi, \lambda)=(0,1)}$ as a bundle homomorphism $\pi_1^* F_0 \rightarrow \pi_1^* F_1$, in (3) as a bundle homomorphism $F_0 \rightarrow F_1$ that induces a map between the L_2 -spaces).

Then there exist a $B(\lambda) \in \mathbf{L}_{\text{w-cl}}^0(\Lambda; M, E_1, E_0)$ and an $R(\lambda) \in L_{\text{cl}}^{-\mu}(\Lambda; M, E_1, E_1)$ such that $P_0 B(\lambda) R(\lambda) P_1$ is the inverse of $P_1 A(\lambda) P_0$ for sufficiently large λ .⁶

PROOF. Let $R(\lambda) \in L_{\text{cl}}^{-\mu}(\Lambda; M, F_1, F_1)$ be a reduction of orders, i.e., there exists an $S(\lambda) \in L_{\text{cl}}^{\mu}(\Lambda; M, F_1, F_1)$ such that $S(\lambda)R(\lambda) = R(\lambda)S(\lambda) = 1$ for all $\lambda \in \Lambda$. Then

$$\tilde{A}(\lambda) := R(\lambda)A(\lambda) \in L_{\text{cl}}^0(\Lambda; M, F_0, F_1) \subset \mathbf{L}_{\text{w-cl}}^0(\Lambda; M, F_0, F_1)$$

and, due to Lemma 5.1,

$$\tilde{P}_1(\lambda) := R(\lambda)P_1S(\lambda) \in \mathbf{L}_{\text{w-cl}}^0(\Lambda; M, F_1, F_1).$$

Obviously, $\tilde{P}_1(\lambda)$ is a projection. The assumptions (1), (2) and (3) in Theorem 5.2 now imply that

$$(5.2) \quad \tilde{P}_1(\lambda)\tilde{A}(\lambda)P_0 \in \mathbf{T}_{\text{w-cl}}^0(\Lambda; M, (F_0, P_0), (F_1, \tilde{P}_1))$$

is a parameter-elliptic Toeplitz operator (the homogeneous principal symbol and the angular symbol are covered by (1) and (2), respectively, while (3) covers the limit-family, cf. Lemma 5.1.b)). Thus we find a parametrix

$$P_0 B(\lambda)\tilde{P}_1(\lambda) \in \mathbf{T}_{\text{w-cl}}^0(\Lambda; M, (F_1, \tilde{P}_1), (F_0, P_0))$$

that coincides with the inverse for large parameter. Then

$$(P_0 B(\lambda)R(\lambda)P_1)(P_1 A P_0) = (P_0 B\tilde{P}_1(\lambda))(\tilde{P}_1(\lambda)\tilde{A}(\lambda)P_0) = P_0$$

for large $|\lambda|$, as well as

$$\begin{aligned} (P_1 A(\lambda)P_0)(P_0 B(\lambda)R(\lambda)P_1) &= S(\lambda)(\tilde{P}_1(\lambda)\tilde{A}(\lambda)P_0)(P_0 B\tilde{P}_1(\lambda))R(\lambda) \\ &= S(\lambda)\tilde{P}_1(\lambda)R(\lambda) = P_1. \end{aligned}$$

Hence $P_0 B(\lambda)R(\lambda)P_1$ is the desired parametrix. \square

5.2. The resolvent of classical operators in projected subspaces. Let $A \in L_{\text{cl}}^{\mu}(M, F, F)$ be a classical pseudodifferential operator of integer order $\mu > 0$ and $P \in L_{\text{cl}}^0(M, F, F)$ be a pseudodifferential projection. We consider the densely defined, unbounded operator

$$(5.3) \quad A_P := PAP : \mathcal{C}^{\infty}(M, F, P) \subset H^s(M, F, P) \longrightarrow H^s(M, F, P),$$

where $s \in \mathbb{R}$ is arbitrary and $\mathcal{C}^{\infty}(M, F, P) := P(\mathcal{C}^{\infty}(M, F))$, and want to study its closedness as well as the existence and the structure of its resolvent.

Let us define

$$A(\lambda) = \tau^{\mu} e^{i\theta} - A, \quad \lambda \in \Lambda,$$

⁶To be precise: If $A'(\lambda) := P_1 A(\lambda)P_0$ and $B'(\lambda) := P_0 B(\lambda)R(\lambda)P_1$ then both $B'(\lambda)A'(\lambda) - P_0 \in \mathbf{L}_0^{-\infty}(\Lambda; M, (F_0, P_0), (F_0, P_0))$ and $A'(\lambda)B'(\lambda) - P_1 \in \mathbf{L}_0^{-\infty}(\Lambda; M, (F_1, P_1), (F_1, P_1))$ vanish for large $|\lambda|$.

and let $R(\lambda) \in L_{\text{cl}}^{-\mu}(\Lambda; M, F, F)$ be a reduction of orders with inverse $S(\lambda) \in L_{\text{cl}}^{\mu}(\Lambda; M, F, F)$. Moreover, let $a^{(\mu)}$, $r^{(-\mu)}$ and $p^{(0)}$ denote the homogeneous principal symbol of A , $R(\lambda)$ and P , respectively.

THEOREM 5.3. *Assume that*

$$p^{(0)}(\tau^{\mu}e^{i\theta} - a^{(\mu)})p^{(0)} : E(P) \longrightarrow E(P), \quad E(P) = p^{(0)}(\pi_1^*F),$$

is an isomorphism for any $\lambda = (\tau, \theta) \in \Lambda$. Then there exists a $B(\lambda) \in \mathbf{L}_{\text{w-cl}}^0(\Lambda; M, F, F)$ such that $PB(\lambda)R(\lambda)P$ is the inverse of $PA(\lambda)P$ for sufficiently large λ .⁷

PROOF. The proof is very similar to that of Theorem 5.2; the main difference is that $A(\lambda)$ has a special form and, in general, does not belong to $L_{\text{cl}}^{\mu}(\Lambda; M, F, F)$ (this is only true when A is a differential operator). Let us now define

$$\tilde{A}(\lambda) := R(\lambda)A(\lambda) = R(\lambda)\tau^{\mu}e^{i\theta} - R(\lambda)A.$$

Then $\tilde{A}(\lambda)$ belongs to $\mathbf{L}_{\text{w-cl}}^0(\Lambda; M, F, F)$ and has homogeneous principal symbol

$$\tilde{a}^{(0)} = r^{(-\mu)}(\tau^{\mu}e^{i\theta} - a^{(\mu)}).$$

Correspondingly, cf. (4.10), the angular symbol equals $r^{(-\mu)}|_{(\xi, \tau)=(0,1)}e^{i\theta}$. For the limit-family observe first that $R(\lambda)\tau^{\mu}e^{i\theta} \in L_{\text{cl}}^0(\Lambda; M, F, F)$. Hence its limit-family is $r^{(-\mu)}|_{(\xi, \tau)=(0,1)}e^{i\theta}$. From Lemma 5.1.a) (with $r_0 = 1$ and $\mu_1 = -\mu$) we conclude that $R(\lambda)\tilde{A}$ has vanishing limit-family.

Defining the projection $\tilde{P}(\lambda) = R(\lambda)PS(\lambda)$ and using Lemma 5.1 we then see that the assumption of the theorem implies that $\tilde{P}(\lambda)\tilde{A}(\lambda)P$ is a parameter-elliptic element of $\mathbf{L}_{\text{w-cl}}^0(\Lambda; M, (F, \tilde{P}), (F, P))$. It therefore has a parametrix of the form $PB(\lambda)\tilde{P}(\lambda)$ that coincides with the inverse for large $|\lambda|$. The theorem follows. \square

With help of this theorem it is easy to derive as a corollary the following result on the operator A_P . Given the strip Λ let us write

$$\Lambda^{\wedge} = \{z = \tau e^{i\theta} \mid (\tau, \theta) \in \Lambda\}.$$

THEOREM 5.4. *Let $a^{(\mu)}$ and $p^{(0)}$ be the homogeneous principal symbol of A and P , respectively. If*

$$(5.4) \quad p^{(0)}a^{(\mu)}p^{(0)} : E(P) \longrightarrow E(P), \quad E(P) = p^{(0)}(\pi_1^*F),$$

has (fibrewise) no spectrum in Λ^{\wedge} , then A_P has a unique closed extension, given by the action of A_P on the domain $H^{s+\mu}(M, F, P)$. Denoting this extension again by A_P , the resolvent $(z - A_P)^{-1}$ exists for sufficiently large $z \in \Lambda^{\wedge}$ and satisfies the uniform estimate

$$\|(z - A_P)^{-1}\|_{\mathcal{L}(H^s(M, F, P))} \leq C|z|^{-1}$$

with some constant C .

⁷Compare the previous footnote for a more detailed statement.

In fact it is clear that $H^{s+\mu}(M, F, P)$ belongs to the domain of the closure of A_P . Now let u belong to the maximal domain, i.e., both u and $A_P u = P A P u$ belong to $H^s(M, F, P)$. With the notation of Theorem 5.3 it follows that

$$u = PB(\lambda)R(\lambda)P(PA(\lambda)Pu) \in PB(\lambda)R(\lambda)P(H^s(M, F, P)) \subset H^{s+\mu}(M, F, P).$$

Hence the closure and the maximal closed extension coincide and have domain $H^{s+\mu}(M, F, P)$. The stated norm estimate follows by writing $z = \tau^\mu e^{i\theta}$ and using that then

$$\|(z - A_P)^{-1}\|_{\mathcal{L}(H^s(M, F, P))} = \|PB(\tau, \theta)R(\tau, \theta)P\|_{\mathcal{L}(H^s(M, F, P))} \leq C|\tau|^{-\mu} = C|z|^{-1}$$

provided $|z|$ is large enough (the estimate holds true, since $R(\lambda) \in L_{\text{cl}}^{-\mu}(\Lambda; M, F, F)$).

REMARK 5.5. *With the above notation choose an elliptic operator $B \in L_{\text{cl}}^\mu(M, F, F)$ having scalar principal symbol $-|\xi|^\mu$. Then we have*

$$P(\tau^\mu - A)P + (1 - P)(\tau^\mu - B)(1 - P) = \tau^\mu - C$$

with

$$C = PAP + (1 - P)B(1 - P) \in L_{\text{cl}}^\mu(M, F, F).$$

Now $c^{(\mu)} : \pi_1^* F \rightarrow \pi_1^* F$, the homogeneous principal symbol of C , does not have spectrum in Λ^\wedge if and only if this is true for (5.4). In this case we can apply the calculus of [Gru85] to obtain a parametrix $D(\tau)$ which coincides with $(\tau^\mu - C)^{-1}$ for large τ . Then $PD(\tau)P$ is the inverse of $P(\tau^\mu - A)P$ for large τ .

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